

Structure of dualities in bosonic string theory

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ABSTRACT

The nature of duality symmetries is explored in closed bosonic string theory, particularly in the case of a four-dimensional target space admitting a one-parameter isometry. It appears that the S-duality of string theory behaves analogously to the Ehlers' symmetry of General Relativity. Furthermore, it is demonstrated that the $O(1,1)$ target space duality arising from the isometry interchanges the roles of these two symmetries. The inclusion of the tachyon field is shown to be consistent with T-duality but incompatible with S-duality. Finally, extrapolating to dimensions other than four, the effective action is found to be invariant under a larger group of symmetries than the usual $O(1,1)$.

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1. Introduction

It is becoming increasingly apparent that string theory possesses a far richer structure than is evident from the conventional perturbative formulation, containing many more symmetries than the Standard Model and General Relativity which it will hopefully unite or replace. Two of these symmetries have become known as string dualities, explicitly S-duality and T-duality (target space duality). The former is an $SL(2, R)$ symmetry acting on the equations of motion of the effective four dimensional action of heterotic string theory, which is related to the electric magnetic duality of four dimensional field theory and may lead to a realization of the strong weak duality conjecture of Montonen and Olive [1], under which the weak coupling limit exchanges with the strong coupling limit. If exact, this symmetry would be of immense benefit: from a knowledge of low energy string theory the structure of strongly coupled string theory could be determined, a domain otherwise only accessible with as yet unknown non-perturbative techniques. Unfortunately, it is this non-perturbative behaviour that removes the possibility of proving exactness order by order in perturbation theory. However, there is some evidence that suggests that the duality is exact [2]. As we shall see, S-duality is also related in certain circumstances to the Ehlers' symmetry of General Relativity. T-duality, however, is an entirely new phenomenon, unique to string theory. It is most clearly demonstrated in the simple model of a string moving on a background consisting of a compact dimension of radius R : the theory is invariant under the duality transformation $R \rightarrow \frac{1}{2R}$ (in units such that $\alpha' = 1$). This can be physically understood as an exchange of the string momentum and winding modes. In the case of a target space with d compactified dimensions, this extends to an $O(d, d; Z)$ transformation between two differing sets of target space fields, which preserves the underlying conformal field theory. Unlike S-duality, T-duality has been shown to be exact [3].

So we have: T-duality, which is a perturbatively exact symmetry, acting in the presence of compact target space dimensions; and S-duality, conjectured to be exact, acting in an effective four dimensional target space. It would be interesting to investigate the combined effects of S and T-duality. In [4], S and T-duality were studied in bosonic string theory with two Abelian isometries, where it was shown that a string generalization of the Geroch group is obtained. We shall again look at bosonic string theory, but concentrate on a thorough analysis of the case with just one isometry and shall discover an interesting phenomenon not immediately apparent in the study with two isometries. At the level of this investigation, i.e. looking only at the effective action, it does not matter whether this isometry is compact or not, and it will be assumed that the basic symmetries will be $O(1, 1; R)$ and $SL(2, R)$ for T-duality and S-duality respectively.

The effective action of the target space, \mathcal{M} , is

$$I = \int_{\mathcal{M}} d^4x \sqrt{\tilde{g}} e^{\Phi} \left(\tilde{R}(\tilde{g}) + (\nabla\Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \quad (1)$$

where \tilde{g} is the target space metric, $\tilde{R}(\tilde{g})$ is the corresponding Ricci scalar, Φ is the dilaton, and

$H_{\mu\nu\rho}$ is the potential of the Kalb-Ramond antisymmetric tensor $B_{\mu\nu}$,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} . \quad (2)$$

The tachyon field has been set to zero for this analysis, but we will study the effect of its inclusion in a later section. Because the target space considered here is not of the critical dimension ($d = 26$ for the bosonic theory), the total central charge must be made up to zero. Including in the action a cosmological constant type term, as in [5], will be shown later to be incompatible with S-duality of the effective action. Instead, we will assume that the effective four dimensional theory is tensored with an appropriate conformal field theory of the required central charge, that can be ignored for the purposes of this analysis. Alternatively, (1) can be considered as a truncation of critical bosonic or super/heterotic string theory, compactified on a suitable manifold. This action actually represents the maximal sector that is common to all of the standard string theories: bosonic, superstring types I, IIA, IIB; and heterotic. The metric \tilde{g} will be taken positive definite, though the entire analysis goes through for any choice of signature. By the conformal rescaling $\tilde{g}_{\mu\nu} = e^{-\Phi} g_{\mu\nu}$, (1) can be rewritten in the Einstein frame

$$I = \int_{\mathcal{M}} d^4x \sqrt{g} \left(R(g) - \frac{1}{2} (\nabla\Phi)^2 - \frac{1}{12} e^{2\Phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) , \quad (3)$$

where all indices are now raised and lowered with g . For the existence of the isometry, there can be no boundaries in \mathcal{M} normal to the isometry. It will be further assumed that \mathcal{M} is compact, though this condition may be relaxed with a proper treatment of surface terms.

Note that the following non-standard normalization will be used for symmetrization and anti-symmetrization of tensors:

$$A_{(\mu_1 \dots \mu_n)} = \frac{1}{(n-1)!} (A_{\mu_1 \dots \mu_n} + \text{cyclic perms} + \text{anticyclic perms}) \quad (4)$$

$$A_{[\mu_1 \dots \mu_n]} = \frac{1}{(n-1)!} (A_{\mu_1 \dots \mu_n} + \text{cyclic perms} - \text{anticyclic perms}) . \quad (5)$$

This particular normalization enables the following definitions to take the form

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad (6)$$

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} . \quad (7)$$

Also, $\epsilon_{\mu\nu\rho\sigma}$ and ϵ_{ijk} , represent the totally antisymmetric tensors in four and three dimensions respectively.

2. S-Duality of the effective action

A simplified version of S-duality exists for bosonic string theory in an effective four-dimensional background; it is simplified as there are no Maxwell gauge fields, and hence the electric-magnetic

duality is absent. The remaining symmetry acts upon the fields appearing in the action (1). It is interesting to note that the effect of S-duality decouples from the metric field precisely when the metric is conformally rescaled to the Einstein frame. The importance of this point will be seen later. For now this observation simply leads us to work with the Einstein frame action (3) for ease of computations.

There appears to be some confusion in the literature with regards to the method for constructing the manifestly S-duality invariant action from (3). The 3-form, $H_{\mu\nu\rho}$, is usually dualized by writing

$$H_{\mu\nu\rho} = e^{-2\Phi} \epsilon_{\mu\nu\rho}{}^\sigma \partial_\sigma \Psi \quad (8)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor and Ψ is the axion pseudo-scalar field. Substituting (8) into the action gives

$$I = \int_{\mathcal{M}} d^4x \sqrt{g} \left(R(g) - \frac{1}{2} (\nabla\Phi)^2 - \frac{1}{2} e^{-2\Phi} (\nabla\Psi)^2 \right) . \quad (9)$$

However, the equations of motion derived from (9) do not agree with those from (3). To obtain an action consistent with (3) we must turn to the language of forms. Let H_3 be the three form $H_{\mu\nu\rho}$, H_1 its Hodge dual, and B_2 the two-form $B_{\mu\nu}$. Now $H_3 = dB_2$ so $dH_3 = 0$, and $H_3 = *H_1$ so $d*H_1 = 0$, but not $dH_1 = 0$ which is essentially the assumption made in (8). Finally if $d*H_1 = 0$ then $*d*H_1 = 0$. This last relation can be written in tensor notation as

$$\epsilon^{\mu\nu\rho\sigma} \nabla_\mu \epsilon_{\nu\rho\sigma}{}^\tau H_\tau = 0 , \quad (10)$$

where H_τ is the Hodge dual of $H_{\mu\nu\rho}$, given by

$$H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}{}^\tau H_\tau . \quad (11)$$

Equation (10) simplifies to

$$\nabla_\mu H^\mu = 0 . \quad (12)$$

The procedure is now clear: we substitute (11) into (3) and add the constraint (12) multiplied by a Lagrange multiplier Ψ

$$I = \int_{\mathcal{M}} d^4x \sqrt{g} \left(R(g) - \frac{1}{2} (\nabla\Phi)^2 - \frac{1}{2} e^{2\Phi} H_\mu H^\mu + \Psi \nabla_\mu H^\mu \right) . \quad (13)$$

The equation of motion for H_μ is

$$e^{2\Phi} H_\mu = -\nabla_\mu \Psi . \quad (14)$$

After inserting (14), integrating by parts, and removing the surface term (13) becomes

$$I = \int_{\mathcal{M}} d^4x \sqrt{g} \left(R(g) - \frac{1}{2} \left((\nabla\Phi)^2 - e^{-2\Phi} (\nabla\Psi)^2 \right) \right) , \quad (15)$$

where Ψ is now the axion pseudo-scalar field. There is an obvious sign difference in the axion term between (9) and (15) and the latter does indeed lead to the same equations of motion as

the original action (3). The dilaton-axion term forms the Lagrangian of an $O(2,1)$ non-linear σ model which is invariant under three transformations:

dilations

$$\begin{aligned}\Phi &\rightarrow \Phi + a \\ \Psi &\rightarrow e^a \Psi ,\end{aligned}\tag{16}$$

translations

$$\begin{aligned}\Phi &\rightarrow \Phi \\ \Psi &\rightarrow \Psi + b ,\end{aligned}\tag{17}$$

and a non-trivial mixing

$$\begin{aligned}e^\Phi &\rightarrow \frac{e^\Phi}{(1 - c\Psi)^2 - c^2 e^{2\Phi}} \\ \Psi &\rightarrow \frac{\Psi + c(e^{2\Phi} - \Psi^2)}{(1 - c\Psi)^2 - c^2 \Psi^2} .\end{aligned}\tag{18}$$

These are more clearly shown by introducing two new scalar fields

$$\lambda_\pm = \Psi \pm e^\Phi\tag{19}$$

and writing (15) as

$$I = \int_{\mathcal{M}} d^4x \sqrt{g} \left(R(g) - \frac{1}{2} \frac{\partial \lambda_+ \partial \lambda_-}{(\lambda_+ - \lambda_-)^2} \right) .\tag{20}$$

The transformations (16)-(18) can now be recognized as an $SL(2, R)$ group of isometries realized by

$$\lambda_\pm \rightarrow \lambda'_\pm = \frac{a\lambda_\pm + b}{c\lambda_\pm + d} , \quad ad - bc = 1 .\tag{21}$$

Obviously (20) is invariant under (21) which forms the S-duality of bosonic string theory. For the heterotic string, S-duality is reduced to a symmetry of the equations of motion of the effective action, and furthermore, it is believed that the $SL(2, R)$ symmetry will be broken down to an $SL(2, Z)$ symmetry by instanton effects [2].

3. S-duality in the presence of an isometry

Anticipating our study of T-duality, it would be useful to examine the behaviour of S-duality when \mathcal{M} contains a one-parameter isometry generated by the Killing vector $K = K^\mu \partial_\mu = \partial_0$. If

the metric were Lorentzian rather than Riemannian, K could be either time-like or space-like, but not null for the following analysis. Thus

$$\mathcal{L}_K(g_{\mu\nu}) = 0, \quad \mathcal{L}_K(B_{\mu\nu}) = 0, \quad \mathcal{L}_K\Phi = 0, \quad (22)$$

where \mathcal{L} is the Lie derivative. The isometry defines a fibering $\pi : \mathcal{M} - \mathcal{C} \rightarrow \Sigma$ where Σ is the three dimensional base manifold and \mathcal{C} is the fixed point set of the isometry. Locally, the metric on \mathcal{M} can be written [6]

$$g_{\mu\nu} = \begin{pmatrix} e^U & e^U \omega_j \\ e^U \omega_i & e^U \omega_i \omega_j + e^{-U} \gamma_{ij} \end{pmatrix}, \quad (23)$$

$$g^{\mu\nu} = \begin{pmatrix} e^U \omega_i \omega_j \gamma^{ij} + e^{-U} & -e^U \omega_i \gamma^{ij} \\ -e^U \omega_j \gamma^{ij} & e^U \gamma^{ij} \end{pmatrix}, \quad (24)$$

where $e^{-U} \gamma_{ij}$ is the d -dimensional metric on Σ , U is a scalar field, and ω_i is a gauge potential defined up to

$$\omega_i \rightarrow \omega_i + \partial_i \chi. \quad (25)$$

The natural projection operator from \mathcal{M} down onto Σ is

$$h^\mu{}_\nu = \delta^\mu{}_\nu - \frac{K^\mu K_\nu}{\|K\|^2}, \quad (26)$$

(note that if g were Lorentzian, and K null, (26) would be ill-defined). A gauge invariant field on Σ can be defined as

$$V_{ij} = \partial_i \omega_j - \partial_j \omega_i. \quad (27)$$

The fields V_{ij} and γ_{ij} are tensors on Σ and their indices are raised and lowered with γ . Using (23) and (24) the curvature tensors on \mathcal{M} can be expressed in terms of U , V_{ij} , γ_{ij} , and their derivatives. In particular, the Ricci scalar becomes

$$R(g) = e^U R(\gamma) + e^U D^2 U - \frac{1}{2} e^U (DU)^2 - \frac{1}{4} e^{3U} V_{ij} V^{ij}, \quad (28)$$

where $R(\gamma)$ is the Ricci scalar of the metric γ_{ij} on Σ , and D_i is the covariant derivative induced by γ_{ij} on Σ . All indices are raised and lowered with γ . The determinant of $g_{\mu\nu}$ is

$$\det(g_{\mu\nu}) = e^{-2U} \det(\gamma_{ij}). \quad (29)$$

By (22), the dilaton term in (15) can be simply expressed as

$$g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi = e^U \gamma^{ij} D_i \Phi D_j \Phi. \quad (30)$$

However, the axion needs more careful treatment. From (22)

$$\partial_0 B_{\mu\nu} = 0, \quad \partial_0 H_{\mu\nu\rho} = 0, \quad (31)$$

from which it follows, using (11), that $\partial_0 H_\mu = 0$. This implies by (14) that

$$\partial_0 \partial_\mu \Psi = 0 , \quad (32)$$

but not $\partial_0 \Psi = 0$. $\nabla_\mu \Psi$ can be projected down from \mathcal{M} onto Σ using the projection operator (26)

$$\begin{aligned} \partial_0 \Psi &= K^\mu \nabla_\mu \Psi = \nabla_0 \Psi \\ D_i \Psi &= h^\mu_i \nabla_\mu \Psi = \nabla_i \Psi - \omega_i \nabla_0 \Psi . \end{aligned} \quad (33)$$

$\nabla_\mu \Psi$ can now be written as

$$\nabla_\mu \Psi = \begin{pmatrix} \partial_0 \Psi \\ D_i \Psi + \omega_i \partial_0 \Psi \end{pmatrix} . \quad (34)$$

Thus

$$g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi = e^U \gamma^{ij} D_i \Psi D_j \Psi + e^{-U} (\partial_0 \Psi)^2 . \quad (35)$$

We can substitute (28), (29), (30), and (35) into (15), integrate the $D^2 U$ term by parts, and remove the surface term to obtain

$$I = \int_{\mathcal{M}} d^4 x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2} (DU)^2 - \frac{1}{4} e^{2U} V_{ij} V^{ij} - \frac{1}{2} \left((D\Phi)^2 - e^{-2\Phi} (D\Psi)^2 \right) - \frac{1}{2} e^{-2(\Phi+U)} (\partial_0 \Psi)^2 \right) . \quad (36)$$

The last term of (36) can be written

$$e^{-2(\Phi+U)} (\partial_0 \Psi)^2 = \partial_0 \left(e^{-2(\Phi+U)} \Psi \partial_0 \Psi \right) - e^{-2(\Phi+U)} \partial_0 \partial_0 \Psi . \quad (37)$$

The first term on the rhs of (37) is a vanishing surface term and the second term vanishes by (32). With all dependence on x^0 explicitly removed, the action (36) can be rewritten as an effective three-dimensional action on Σ ,

$$I = \int_{\Sigma} d^3 x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2} (DU)^2 - \frac{1}{4} e^{2U} V_{ij} V^{ij} - \frac{1}{2} \left((D\Phi)^2 - e^{-2\Phi} (D\Psi)^2 \right) \right) , \quad (38)$$

where all indices are now raised and lowered with γ . The Hodge dual of V_{ij} is V_i given by

$$V_{ij} = \epsilon_{ij}{}^k V_k , \quad (39)$$

where V_i obeys the conservation equation

$$D_i V^i = 0 , \quad (40)$$

As before, a new action is defined by adding the constraint (40), multiplied by a Lagrange multiplier V , to (38)

$$I = \int_{\Sigma} d^3 x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2} (DU)^2 - \frac{1}{2} e^{2U} V_i V^i + V D_i V^i - \frac{1}{2} \left((D\Phi)^2 - e^{-2\Phi} (D\Psi)^2 \right) \right) . \quad (41)$$

The equation of motion for V_i is

$$e^{2U} V_i = -D_i V . \quad (42)$$

Substituting (42) into (41), integrating by parts, and removing the surface term gives the final form of the action,

$$I = \int_{\Sigma} d^3x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2} \left((DU)^2 - e^{-2U} (DV)^2 \right) - \frac{1}{2} \left((D\Phi)^2 - e^{-2\Phi} (D\Psi)^2 \right) \right) . \quad (43)$$

This action describes a three dimensional spacetime with metric γ_{ij} , scalars U and Φ , and pseudo-scalars V and Ψ . There now appears to be two $O(2, 1)$ non-linear σ -models, each with its own $SL(2, R)$ isometry: the original S-duality and a new isometry which is actually the $SL(2, R)$ Ehlers' group of symmetries from dimensionally reduced General Relativity [6]. So there is a new duality of the four dimensional string effective action with one killing symmetry, and we have termed this E-duality. Obviously the symmetries of this action are larger than the combined S-duality and $O(1, 1)$ target space duality: two independent $SL(2, R)$ symmetries given by (16)-(18) and the same transformations with U, V replacing Φ, Ψ respectively; and (43) is also invariant under

$$\begin{aligned} U &\leftrightarrow \Phi \\ V &\leftrightarrow \Psi . \end{aligned} \quad (44)$$

This final transformation is highly suggestive of discrete target space duality. The total symmetry here is

$$SL(2, R) \otimes SL(2, R) \simeq O(2, 2; R) , \quad (45)$$

though of course this $O(2, 2)$ symmetry is totally distinct to the T-duality $O(2, 2)$ symmetry when there are two isometries. The task is now to find an explicit realization in terms of field redefinitions of the $O(1, 1)$ target space duality under which (43) should be invariant, and to see how this mixes with the duality symmetries found above.

4. T-duality of the effective action

Unlike the case of S-duality, the dimension of the target space is not critical to T-duality; the only requirement is the presence of an isometry. For this reason the following analysis will be performed in arbitrary dimension,

specializing to four dimensions later. We return to the original string frame action (1), now generalized to D-dimensions. Once again, we assume the existence of a one-parameter isometry generated by the Killing vector K , and now Σ is the $d = D - 1$ dimensional base manifold. The procedure in (22)-(28) follows and the string frame Ricci scalar is

$$\tilde{R}(\tilde{g}) = e^U \tilde{R}(\tilde{\gamma}) + (d - 2) e^U D^2 U - \frac{1}{4} (d^2 - 5d + 8) e^U (DU)^2 - \frac{1}{4} e^{3U} V_{ij} V^{ij} , \quad (46)$$

and the determinant of $\tilde{g}_{\mu\nu}$ is

$$\det(\tilde{g}_{\mu\nu}) = e^{U(1-d)} \det(\tilde{\gamma}_{ij}) . \quad (47)$$

The dilaton term is given again by (30), but the reduction of the Kalb-Ramond field is more involved. Using the projection operator (26), $B_{\mu\nu}$ is decomposed into

$$\begin{aligned} b &= K^\mu K^\nu B_{\mu\nu} = 0 \\ b_i &= K^\mu h^\nu_i B_{\mu\nu} = B_{0i} \\ b_{ij} &= h^\mu_i h^\nu_j B_{\mu\nu} = B_{ij} - \omega_i B_{0j} - \omega_j B_{i0} , \end{aligned} \quad (48)$$

and can be written in the form

$$B_{\mu\nu} = \begin{pmatrix} 0 & b_j \\ -b_i & b_{ij} + \omega_{[i} b_{j]} \end{pmatrix} . \quad (49)$$

An analogue of V_{ij} can be constructed for b_i ,

$$W_{ij} = \partial_i b_j - \partial_j b_i . \quad (50)$$

Now $H_{\mu\nu\rho}$ is decomposed into

$$H_{0ij} = -W_{ij} \quad (51)$$

$$H_{ijk} = \partial_{[i} b_{jk]} + b_{[i} V_{jk]} - \omega_{[i} W_{jk]} \quad (52)$$

and the H field term in (1) is written

$$H_{\mu\nu\rho} H^{\mu\nu\rho} = 3e^U W_{ij} W^{ij} + e^{3U} \theta_{ijk} \theta^{ijk} , \quad (53)$$

where

$$\theta_{ijk} = \partial_{[i} b_{jk]} + b_{[i} V_{jk]} . \quad (54)$$

Inserting (29), (28), (30), and (53) in (1), the action becomes independent of x^0 and reduces to

$$\begin{aligned} I = \int_{\Sigma} d^d x \sqrt{\tilde{\gamma}} e^{\Phi + \frac{1}{2}U(3-d)} & \left(\tilde{R}(\tilde{\gamma}) + (d-2)D^2 U - \frac{1}{4}(d^2 - 5d + 8)(DU)^2 + (D\Phi)^2 \right. \\ & \left. - \frac{1}{4}e^{2U} V_{ij} V^{ij} - \frac{1}{4}W_{ij} W^{ij} - \frac{1}{12}e^{2U} \theta_{ijk} \theta^{ijk} \right) . \end{aligned} \quad (55)$$

Following [7], (55) should be invariant under the $O(1,1)$ duality symmetry, consisting of a one-parameter scaling

$$\begin{aligned} \tilde{g}_{00} &\rightarrow e^a \tilde{g}_{00} , \\ \tilde{g}_{0i} &\rightarrow e^{a/2} \tilde{g}_{0i} , \\ B_{0i} &\rightarrow e^{a/2} B_{0i} , \end{aligned} \quad (56)$$

and a discrete inversion

$$\begin{aligned}
\tilde{g}_{00} &\rightarrow 1/\tilde{g}_{00} , \\
\tilde{g}_{0\alpha} &\rightarrow B_{0\alpha}/\tilde{g}_{00} , \\
\tilde{g}_{\alpha\beta} &\rightarrow \tilde{g}_{\alpha\beta} - (\tilde{g}_{0\beta}\tilde{g}_{0\alpha} - B_{0\alpha}B_{0\beta})/\tilde{g}_{00} , \\
B_{0\alpha} &\rightarrow \tilde{g}_{0\alpha}/\tilde{g}_{00} , \\
B_{\alpha\beta} &\rightarrow B_{\alpha\beta} - \tilde{g}_{0[\beta}B_{\alpha]0}/\tilde{g}_{00} .
\end{aligned} \tag{57}$$

A dilaton transformation is required in both these transformations to ensure conformal invariance is preserved in the dual theory (at one loop). This is

$$\phi \rightarrow \phi + \ln \tilde{g}_{00} . \tag{58}$$

The explicit form of the symmetry can be rewritten in terms of redefinitions of the fields appearing in the action (55). The scaling can be realized in the following field redefinitions:

$$\begin{aligned}
U &\rightarrow U + a \\
\omega_i &\rightarrow e^{-a/2}\omega_i \\
b_i &\rightarrow e^{a/2}b_i \\
\tilde{\gamma}_{ij} &\rightarrow e^a\tilde{\gamma}_{ij} \\
\Phi &\rightarrow \Phi - a/2 ,
\end{aligned} \tag{59}$$

under which the target space fields become

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} e^{U+a} & e^{U+a/2}\omega_j \\ e^{U+a/2}\omega_i & e^U\omega_i\omega_j + e^{-U}\tilde{\gamma}_{ij} \end{pmatrix} , \quad B_{\mu\nu} = \begin{pmatrix} 0 & e^{a/2}b_j \\ -e^{a/2}b_i & b_{ij} + \omega_{[i}b_{j]} \end{pmatrix} . \tag{60}$$

The inversion part of the symmetry is similarly realised as

$$\begin{aligned}
U &\rightarrow -U \\
\omega_i &\rightarrow b_i \\
b_i &\rightarrow \omega_i \\
\tilde{\gamma}_{ij} &\rightarrow e^{-2U}\tilde{\gamma}_{ij} \\
b_{ij} &\rightarrow b_{ij} + \omega_i b_j - \omega_j b_i \\
\Phi &\rightarrow \Phi + U ,
\end{aligned} \tag{61}$$

under which the target space fields become

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} e^{-U} & e^{-U}b_j \\ e^{-U}b_i & e^{-U}b_i b_j + e^{-U}\tilde{\gamma}_{ij} \end{pmatrix} , \quad B_{\mu\nu} = \begin{pmatrix} 0 & \omega_j \\ -\omega_i & b_{ij} \end{pmatrix} . \tag{62}$$

It can be shown that (55) is indeed invariant under field redefinitions (59) and (61). Furthermore, the d-dimensional metric $e^{-U}\tilde{\gamma}_{ij}$, and the d-dimensional 3-form θ_{ijk} are both invariant under these redefinitions.

We can see from the field redefinitions in (61) that the discrete duality symmetry involves exchanging the V_{ij} and W_{ij} in (55). It would be interesting to see whether (55) could be put in a more appropriate form to examine this field exchange. First the D^2U term is removed by integrating by parts then removing the total divergence, giving

$$I = \int_{\Sigma} d^d x \sqrt{\tilde{\gamma}} e^{\Phi + \frac{1}{2}U(3-d)} \left(\tilde{R}(\tilde{\gamma}) + \frac{1}{4}(d-1)(d-4)(DU)^2 - (d-2)DU D\Phi + (D\Phi)^2 \right. \\ \left. - \frac{1}{4}e^{2U}V_{ij}V^{ij} - \frac{1}{4}W_{ij}W^{ij} - \frac{1}{12}e^{2U}\theta_{ijk}\theta^{ijk} \right). \quad (63)$$

This action can be transformed to the Einstein frame with respect to $\tilde{R}(\tilde{\gamma})$ by introducing a new metric γ_{ij} where

$$\gamma_{ij} = \exp\left(\frac{2\Phi + U(3-d)}{d-2}\right) \tilde{\gamma}_{ij} \quad (64)$$

and thus (63) becomes (for $d \neq 2$)

$$I = \int_{\Sigma} d^d x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{4}\frac{(d-1)}{(d-2)}(DU)^2 - \frac{1}{(d-2)}DU D\Phi - \frac{1}{(d-2)}(D\Phi)^2 \right. \\ \left. - \frac{1}{4}\exp\left(\frac{2\Phi + U(d-1)}{d-2}\right)V_{ij}V^{ij} - \frac{1}{4}\exp\left(\frac{2\Phi + U(3-d)}{d-2}\right)W_{ij}W^{ij} \right. \\ \left. - \frac{1}{12}\exp\left(\frac{4\Phi + 2U}{d-2}\right)\theta_{ijk}\theta^{ijk} \right). \quad (65)$$

The U and Φ terms can be decoupled by defining new variables \tilde{U} and $\tilde{\Phi}$ given by

$$U = 2(1 + \sqrt{d-2})\tilde{U} - 2(1 + \sqrt{d-2})\tilde{\Phi}, \quad (66)$$

$$\Phi = (d-3)\tilde{U} + (d-1+2\sqrt{d-2})\tilde{\Phi}. \quad (67)$$

Inserting (66) and (67) into (65) gives the most symmetric form of the action,

$$I = \int_{\Sigma} d^d x \sqrt{\gamma} \left(R(\gamma) - c(D\tilde{U})^2 - c(D\tilde{\Phi})^2 \right. \\ \left. - \frac{1}{4}e^{x\tilde{U}+y\tilde{\Phi}}V_{ij}V^{ij} - \frac{1}{4}e^{x\tilde{\Phi}+y\tilde{U}}W_{ij}W^{ij} - \frac{1}{12}e^{(x+y)(\tilde{U}+\tilde{\Phi})}\theta_{ijk}\theta^{ijk} \right) \quad (68)$$

where

$$c = 2(d-1+2\sqrt{d-2}) \quad x = 4 + 2\frac{d-1}{\sqrt{d-2}} \quad y = 2\frac{3-d}{\sqrt{d-2}}. \quad (69)$$

The $O(1,1)$ duality transformations (59), (61) can be rewritten in terms of the new variables defined in (64), (66), and (67). The scaling part becomes

$$\begin{aligned}
\tilde{U} &\rightarrow \tilde{U} + a \\
\tilde{\Phi} &\rightarrow \tilde{\Phi} - a \\
V_{ij} &\rightarrow e^{a(y-x)/2} V_{ij} \\
W_{ij} &\rightarrow e^{a(x-y)/2} W_{ij} \\
\theta_{ijk} &\rightarrow \theta_{ijk} \\
\gamma_{ij} &\rightarrow \gamma_{ij} ,
\end{aligned} \tag{70}$$

and the inversion,

$$\begin{aligned}
\tilde{U} &\leftrightarrow \tilde{\Phi} \\
V_{ij} &\leftrightarrow W_{ij} \\
\theta_{ijk} &\rightarrow \theta_{ijk} \\
\gamma_{ij} &\rightarrow \gamma_{ij} .
\end{aligned} \tag{71}$$

The discrete inversion of the $O(1,1)$ transformation is now explicitly shown in terms of a simple field exchange symmetry.

Returning to the case of $d = 3$ ($D = 4$), the action (68), after rescaling $\tilde{\Phi}$ and \tilde{U} by a factor of four, becomes

$$I = \int_{\Sigma} d^3x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2}(D\tilde{U})^2 - \frac{1}{2}(D\tilde{\Phi})^2 - \frac{1}{4}e^{2\tilde{U}}V_{ij}V^{ij} - \frac{1}{4}e^{2\tilde{\Phi}}W_{ij}W^{ij} - \frac{1}{12}e^{2\tilde{U}}e^{2\tilde{\Phi}}\theta_{ijk}\theta^{ijk} \right) . \tag{72}$$

The 3-form θ_{ijk} in three dimensions must be some scalar, θ , multiplied by the volume form,

$$\theta_{ijk} = \theta \epsilon_{ijk} . \tag{73}$$

Thus, an equivalent action to (72) would be that action with the $\theta_{ijk}\theta^{ijk}$ term replaced by

$$\mathcal{L}_{\theta} = -\frac{1}{2}e^{2\tilde{U}}e^{2\tilde{\Phi}}\theta^2 + \frac{1}{6}\Lambda\epsilon_{ijk}\left(\theta\epsilon^{ijk} - \partial^{[i}b^{jk]} - b^{[i}V^{jk]}\right) , \tag{74}$$

where the equation of motion for the Lagrange multiplier, Λ , gives back the definition of θ_{ijk} as a constraint (54). θ can be consistently integrated out of (74) via its equation of motion,

$$e^{2\tilde{U}}e^{2\tilde{\Phi}}\theta = \Lambda , \tag{75}$$

to give the first order form for \mathcal{L}_{θ} ,

$$\mathcal{L}_{\theta} = \frac{1}{2}e^{-2(\tilde{U}+\tilde{\Phi})}\Lambda^2 - \frac{1}{6}\Lambda\epsilon_{ijk}\left(\partial^{[i}b^{jk]} + b^{[i}V^{jk]}\right) . \tag{76}$$

The equation of motion for b_{ij} ,

$$\partial_i \Lambda = 0 , \quad (77)$$

reveals that Λ is in fact a constant, not a scalar [8], and the equation of motion for Λ ,

$$e^{-2(\tilde{U}+\tilde{\Phi})} \Lambda = \frac{1}{6} \epsilon_{ijk} \left(\partial^{[i} b^{jk]} + b^{[i} V^{jk]} \right) , \quad (78)$$

enables us to write

$$\mathcal{L}_\theta = -\frac{1}{2} e^{-2(\tilde{U}+\tilde{\Phi})} \Lambda^2 . \quad (79)$$

However, using (52) in (78) gives

$$e^{-2(\tilde{U}+\tilde{\Phi})} \Lambda = \frac{1}{6} e^{-3\tilde{U}} \epsilon_{ijk} H^{ijk} , \quad (80)$$

and recalling the relations (11), (14), and (34) we obtain

$$\mathcal{L}_\theta = -\frac{1}{2} e^{-2(\tilde{U}+\tilde{\Phi})} (\partial_0 \Psi)^2 . \quad (81)$$

As before, integrating this by parts (37) and using (32) shows that \mathcal{L}_θ vanishes. In other words, the cosmological term that should arise when there is a quadratic n-form term in n dimensions [8] actually vanishes in this particular case. The 2-forms V_{ij} and W_{ij} may be dualized using the procedure set out in (39)-(43) into the pseudo-scalars V and Ψ respectively and the final form of the action becomes

$$I = \int_\Sigma d^3x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2} \left((D\tilde{U})^2 - e^{-2\tilde{U}} (DV)^2 \right) - \frac{1}{2} \left((D\tilde{\Phi})^2 - e^{-2\tilde{\Phi}} (D\Psi)^2 \right) \right) , \quad (82)$$

which is identical to (43). The explicit action of $O(1,1)$ duality upon (43) is now known and is given by the following field redefinitions

$$\begin{aligned} \tilde{U} &\rightarrow \tilde{U} + a \\ \tilde{\Phi} &\rightarrow \tilde{\Phi} - a \\ V &\rightarrow e^{-a} V \\ \Psi &\rightarrow e^a \Psi , \end{aligned} \quad (83)$$

and

$$\begin{aligned} \tilde{U} &\leftrightarrow \tilde{\Phi} \\ V &\leftrightarrow \Psi . \end{aligned} \quad (84)$$

The scaling of U and V is already part of the E-duality transformations, and the scaling of Φ and Ψ is already part of the S-duality transformations. It appears that $O(1,1)$ duality simply interchanges the two $SL(2, R)$ symmetries. Thus, there is a very extensive symmetry of the four dimensional bosonic effective action with one isometry, which can be represented by

$$\begin{array}{ccc} \textbf{E-duality} & \longleftrightarrow & \textbf{S-duality} \\ SL(2, R) & O(1, 1) & SL(2, R) \\ & \textbf{T-duality} & \end{array}$$

and the total group structure is thus $O(2, 2; R)$. In [9] it has been shown that heterotic string theory compactified on a 7-torus has a duality group of $O(8, 24; Z)$ which consists of $O(7, 23; Z)$ T-duality and $SL(2, Z)$ S-duality. This is a generalization of the above result.

5. More general bosonic actions

Bosonic string theory suffers from a tachyonic ground state. Fortunately, in the construction of more realistic string theories, this state is projected out. In the above, we have assumed that the tachyon field has been set to zero, which is consistent with treating the action as a truncation of a supersymmetric theory. However, for completeness, we will discuss the case of a non-trivial tachyon. The modified 4-d effective action, generalizing (1), is

$$I = \int_{\mathcal{M}} d^4x \sqrt{\tilde{g}} e^{\Phi} \left(\tilde{R}(\tilde{g}) + (\nabla\Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - (\nabla T)^2 + V(T) \right), \quad (85)$$

where T is the tachyon and $V(T)$ is the tachyon potential [10]. Transforming (85) to the form in which S-duality is manifest, we find

$$I = \int_{\mathcal{M}} d^4x \sqrt{g} \left(R(g) - \frac{1}{2} \{ (\nabla\Phi)^2 - e^{-2\Phi} (\nabla\Psi)^2 \} - (\nabla T)^2 + e^{-\Phi} V(T) \right). \quad (86)$$

Thus, while the inclusion of the tachyon kinetic term is trivially compatible with S-duality, the potential cannot transform so as to keep the last term of (86) invariant. Thus, the tachyon potential explicitly breaks S-duality. Proceeding as before and reducing (86) to an effective 3-d action gives

$$I = \int_{\Sigma} d^3x \sqrt{\gamma} \left(R(\gamma) - \frac{1}{2} ((DU)^2 - e^{-2U} (DV)^2) - \frac{1}{2} ((D\Phi)^2 - e^{-2\Phi} (D\Psi)^2) - (DT)^2 + e^{-(U+\Phi)} V(T) \right) \quad (87)$$

So the potential also explicitly breaks E-duality. However, (87) is invariant under the T-duality scaling and inversion symmetries (83) and (84), with T and $V(T)$ remaining unchanged. This is to be expected as including the tachyon term in the worldsheet action [10] does not affect the worldsheet duality transformation [7]. In arbitrary dimension, the effective action is

$$I = \int_{\Sigma} d^d x \sqrt{\gamma} \left(R(\gamma) - c(D\tilde{U})^2 - c(D\tilde{\Phi})^2 - (DT)^2 + e^{z(U+\Phi)} V(T) \right. \\ \left. - \frac{1}{4} e^{x\tilde{U}+y\tilde{\Phi}} V_{ij} V^{ij} - \frac{1}{4} e^{x\tilde{\Phi}+y\tilde{U}} W_{ij} W^{ij} - \frac{1}{12} e^{(x+y)(\tilde{U}+\tilde{\Phi})} \theta_{ijk} \theta^{ijk} \right) \quad (88)$$

with c , x , y given by (69) and

$$z = 4 - 2d - 2\sqrt{d-2}. \quad (89)$$

A cosmological constant term, Λ , may be added to the original action (1) to give a $c = 0$ string theory with a target space of dimension less than the critical dimension [5],

$$I = \int_{\mathcal{M}} d^d x \sqrt{\tilde{g}} e^{\Phi} \left(\tilde{R}(\tilde{g}) + (\nabla\Phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \Lambda \right), \quad (90)$$

where

$$\Lambda = \frac{2(d-26)}{3\alpha'} . \quad (91)$$

This term enters the action in the same way as the tachyon potential, and in the precisely the same manner explicitly breaks S and E-duality. It is consistent with $O(1,1)$ T-duality.

6. Duality in $d > 4$

Returning to the original d-dimensional action (55), there is another one-parameter isometry that does not belong to the expected $O(1,1)$ group of T-duality transformations:

$$\begin{aligned} U &\rightarrow U + a \\ \Phi &\rightarrow \Phi - a(3-d)/2 \\ \omega_i &\rightarrow e^{-a}\omega_i \\ b_{ij} &\rightarrow e^{-a}b_{ij} , \end{aligned} \quad (92)$$

and $\tilde{g}_{\mu\nu}$, $B_{\mu\nu}$ become

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} e^{U+a} & e^U\omega_j \\ e^U\omega_i & e^{U-a}\omega_i\omega_j + e^{-(U+a)}\tilde{\gamma}_{ij} \end{pmatrix} , \quad \tilde{B}_{\mu\nu} = \begin{pmatrix} 0 & b_j \\ -b_i & e^{-a}(b_{ij} + \omega_{[i}b_{j]}) \end{pmatrix} . \quad (93)$$

Note that neither $e^{-U}\tilde{\gamma}_{ij}$ nor θ_{ijk} are invariant under (92). When examined, this new symmetry combines with the standard scaling symmetry of $O(1,1)$ duality given in (59) to give a much more general set of scalings,

$$\begin{aligned} \tilde{U} &\rightarrow \tilde{U} + a \\ \tilde{\Phi} &\rightarrow \tilde{\Phi} + b \\ V_{ij} &\rightarrow e^{-(ax+by)/2}V_{ij} \\ W_{ij} &\rightarrow e^{-(ay+bx)/2}W_{ij} \\ b_{ij} &\rightarrow e^{-(a+b)(x+y)/2}b_{ij} \\ \gamma_{ij} &\rightarrow \gamma_{ij} , \end{aligned} \quad (94)$$

of which (70) is the special case of $a = -b$. Thus, the symmetry group of the bosonic effective action with one Killing vector is larger than the target space duality $O(1,1)$ symmetry. At first sight the extra symmetry seems to be a remnant of the two independent $SL(2, R)$ symmetries that exist in four dimensions. If the basic T-duality scaling is removed from (94) then the remaining scaling can be written as a transformation on the original fields of (1)

$$\begin{aligned} \tilde{g}_{\mu\nu} &\rightarrow e^a\tilde{g}_{\mu\nu} \\ B_{\mu\nu} &\rightarrow e^aB_{\mu\nu} \\ \Phi &\rightarrow \Phi + a(1-d)/2 . \end{aligned} \quad (95)$$

The overall effect of (95) is to rescale the string coupling constant, e^Φ . At the level of the worldsheet, this amounts to adding a topological term to the worldsheet action. Note that this extra symmetry is respected by neither the tachyon potential (88) nor the cosmological constant term of the previous section.

7. Conclusion

The combined effects of S and T-duality have been investigated in the context of four dimensional closed bosonic string theory with one compact dimension, where they have been found to form an extended $O(2, 2)$ duality group, incorporating the Ehlers' symmetry of General Relativity. We have found the most symmetric form of the effective action which clearly demonstrates its manifest invariance under S and T-duality, and under the 'new' E-duality.

The interchange between S and E-duality is particularly interesting as it shows that with a one dimensional compactification, the effective non-gravitational fields are in a one-to-one correspondence with the Kaluza-Klein fields derived from the dimensionally reduced metric. The dilaton appears on much the same footing as the modulus field U , despite its very different appearance in the bosonic worldsheet action. It would be interesting to see how this picture develops in the supersymmetric theories and exactly what structures are interchanged there under T-duality.

The inclusion of the tachyon field does not affect T-duality of the effective action, which is to be expected as T-duality is an exact symmetry of bosonic string theory [3]. However, the tachyon potential explicitly breaks S-duality. It would appear that although S-duality is a symmetry of the massless sector of bosonic string theory, it cannot be an exact symmetry. It remains to be seen whether S-duality will survive as an exact symmetry of the supersymmetric string theories.

A further symmetry has been discovered which demonstrates the invariance of the bosonic effective action under a rescaling of the string coupling constant.

After the completion of this work, it was noticed that [11] contains some overlap with the results presented here. I would like to thank my supervisor, Malcolm Perry, for thoughts and guidance during the course of this work. I am also grateful to Lloyd Alty, Gary Gibbons, Neil Lambert, Simon Ross, Edward Teo and Paul Townsend for useful discussions and comments.

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